# Task for a Prestressed Reinforced Concrete Cylinder with External Reinforcement and Cylinder Optimization by Varying the Modulus of Elasticity

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**Abstract.** We consider the calculation of the shell in the case when the stressing reinforcement is located on the outer surface. The solution is carried out numerically by the finite element method. A distribution of the modulus of elasticity is determined for which the maximum stresses are zero throughout the entire cylinder.

**Keywords:** Modulus of elasticity · Prestressed reinforced concrete · Optimization

## 1 Introduction

We know that if we reduce in any area of the body the modulus of elasticity, in this same area there will be a reduction of stress [1–6]. This leads to the fact that when solving optimization problems you can resort to the consideration of inverse problems of elasticity theory, i.e. tasks in which is found the law of distribution of the characteristics of the material in which the stress state is given. Solving an inverse problem, specifying a certain strength criterion, according to which find the laws of change of the mechanical characteristics from the coordinates.

However, using iterative algorithms for optimization, it is possible not to resort to the solution of inverse problems of the theory of elasticity, and instead solve the direct problem at each step. Finite element method opens wide opportunities for solving this problem.

## 2 Materials and Methods

Let us consider the calculation of the shell in the case when the prestressing reinforcement is located on the outer surface. The action of the prestressed reinforcement on a concrete cylinder can be replaced by contact pressure $p_b$ (Fig. 1).


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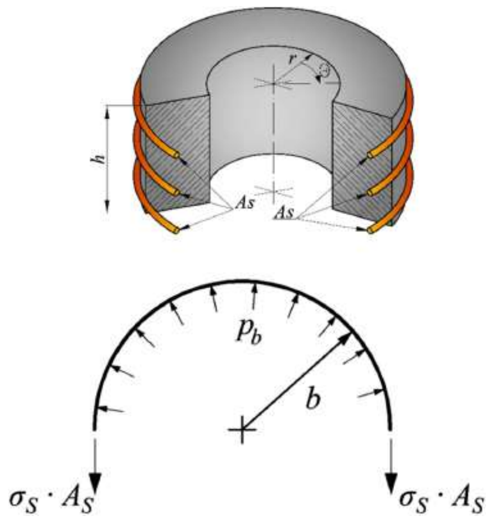


**Fig. 1.** To determining the contact pressure $p_b$.

We assume that the shell is under conditions of plane deformation. The value can be determined from the equilibrium condition of its half

$$2\sigma_S A_S = h\int_{-\pi/2}^{\pi/2} p_b b\cos\varphi d\varphi = 2p_b bh \Rightarrow \sigma_S = \frac{p_b bh}{A_S} \tag{1}$$

The stresses in the annular reinforcement are given by:

$$\sigma_S = E_S \varepsilon_\theta|_{r=b} + \sigma_{sp} \tag{2}$$

Where $\sigma_{sp}$ - the initial stresses in the ring reinforcement (until the moment of transfer of forces to the concrete).

Let the modulus of elasticity of concrete be constant. Then this problem can be solved analytically.

From the solution of the Lame problem for the case of PDS, a formula for displacements [7, 8], $u(r)$:

$$u = \frac{(p_a a^2 - p_b b^2)(1-\nu_1)}{(b^2-a^2)E_{b1}} r - \frac{(p_b - p_a)a^2b^2(1+\nu_1)}{(b^2-a^2)E_{b1}} \frac{1}{r}, \tag{3}$$

where $E_{b1} = \frac{E_b}{1-\nu^2}$, $\nu_1 = \frac{\nu}{1-\nu}$.

Then the expression for the circumferential deformation can be written in the form:

$$\varepsilon_\theta|_{r=b} = \frac{u|_{r=b}}{b} = \frac{(p_a a^2 - p_b b^2)(1-\nu_1)}{(b^2-a^2)E_{b1}} - \frac{(p_b - p_a)a^2(1+\nu_1)}{(b^2-a^2)E_{b1}}, \tag{4}$$

Substituting expressions (1) and (4) into (2), after some transformations we obtain the formula for the contact pressure:

$$p_b = \frac{\frac{2p_a a^2}{(b^2-a^2)E_{b1}} + \frac{\sigma_{sp}}{E_s}}{\frac{bh}{A_s E_s} + \frac{1}{E_{b1}(b^2-a^2)}\left(b^2(1-\nu_1) + a^2(1+\nu_1)\right)}. \tag{5}$$

Further, in order to determine the stresses $\sigma_{b\theta}$ and $\sigma_r$ in concrete, it is necessary to substitute the value $p_b$ in the well-known formulas for the Lamé problem.

Let us find, at what value $A_s$ in the thickness of the cylinder there are no tensile stresses. Stresses $\sigma_{b\theta}$ can be determined by the following formula [1, 9]:

$$\sigma_{b\theta} = -\frac{(p_b - p_a)a^2 b^2}{(b^2-a^2)\cdot r^2} + \frac{p_a a^2 - p_b b^2}{b^2 - a^2} \tag{6}$$

Since the greatest tensile stresses arise on the inner surface, we put in (6) $\sigma_{b\theta}(a) = 0$. Then we obtain the following relation between $p_a$ u $p_b$:

$$p_b = p_a \frac{a^2 + b^2}{2b^2} \tag{7}$$

Expressing from (1) $A_s$, we obtain:

$$A_S = \frac{p_b bh}{\sigma_s} = \frac{p_b bh}{\sigma_{sp} + E_s \varepsilon_\theta|_{r=b}} \tag{8}$$

To find $A_s$, using formula (7), we $p_b$, then using formula (4) calculate $\varepsilon_\theta|_{r=b}$ and substitute the values obtained in (8).

In Fig. 2 shows the stress distribution diagram for $\sigma_{b\theta}$. Calculations were carried out with the following initial data [10–13]:

$$p_a = 10\ \text{MPa},\ a = 1\ \text{m},\ b = 2\ \text{m},\ \nu = 0.2,\ E_b = 2.16 * 10^4,$$
$$E_s = 2 * 10^5,\ \sigma_{sp} = 500\ \text{MPa}.$$

In order to avoid tensile stresses, 2285 kg of reinforcement per 1 m of pipe length were required for the example in question. As can be seen from the graph, $\sigma_{b\theta}$ only for $r = a$, that is the limiting state in this case occurs only at the inner surface.

In the case when the elastic modulus in the thickness is not constant, it is not possible to obtain an analytical solution. Solve the problem numerically, for example, using the finite element method.

We consider the solution of the problem with the help of FEM. We divide the cylinder $n$ along the thickness by one-dimensional finite elements (Fig. 3). The displacements of the nodes of the $i$-th finite element are denoted by $u_i, u_{i+1}$.

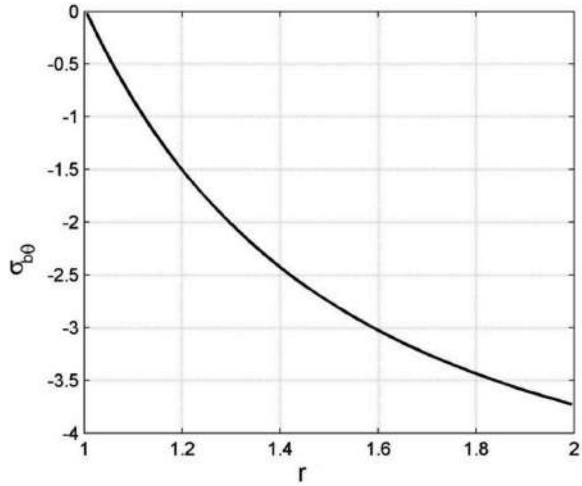


**Fig. 2.** Graph of stress distribution $\sigma_{b\theta}$.

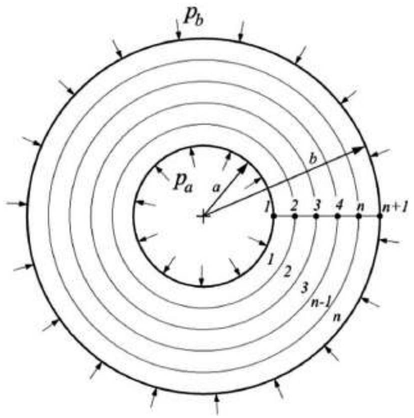


**Fig. 3.** The scheme of partitioning the cylinder into finite elements.

For the stresses in the valve, taking into account the fact that $\varepsilon_\theta = \frac{u}{r}$, we can write:

$$\sigma_S = E_S \varepsilon_\theta|_{r=b} + \sigma_{sp} = E_s \frac{u_{n+1}}{b} + \sigma_{sp}.$$

Then $p_b = \frac{\sigma_s A_s}{bh} = \frac{E_s A_s}{b^2 h} u_{n+1} + \frac{\sigma_{sp} A_s}{bh}$. The work of external forces is defined as follows:

$$A = p_a \cdot 2\pi a \cdot u_1 - p_b \cdot 2\pi b \cdot u_{n+1} = 2\pi \left(p_a \cdot a \cdot u_1 - \frac{E_s A_s}{bh} \cdot u_{n+1}^2 - \frac{\sigma_{sp} A_s}{h} \cdot u_{n+1}\right)$$

After differentiating the work of external forces with respect to nodal displacements, we get:

$$\frac{\partial A}{\partial u_1} = 2\pi a p_a, \frac{\partial A}{\partial u_i} = 0, \textit{if } i \neq 1 \text{ и } i \neq n+1, \frac{\partial A}{\partial u_{n+1}} = 2\pi\left(-2\frac{E_s A_s}{bh} \cdot u_{n+1} - \frac{\sigma_{sp} A_s}{h}\right),$$

Or in the matrix form:

$$\frac{\partial A}{\partial \{U\}} = 2\pi \begin{Bmatrix} p_a \cdot a \\ 0 \\ \dots \\ 0 \\ -\frac{\sigma_{sp} A_s}{h} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -\frac{4\pi E_S A_s}{bh} \end{bmatrix} \cdot \{U\} \tag{9}$$

Thus, in the stiffness matrix obtained when solving a problem for a cylinder loaded with internal and external pressure, the element $K(n+1, n+1)$ must be added to the element $-(4\pi E_S A_s)/(bh)$, and the load vector will be determined by the first term of expression (9).

## 3 Results

Let us find a distribution of the modulus of elasticity for which $\sigma_{b\theta} = 0$ the entire thickness of the cylinder. Consider the equilibrium of its half:

$$2\sigma_S A_S = h \int_{-\pi/2}^{\pi/2} p_a b \cos\varphi d\varphi = 2 p_a b h$$

But from (1):

$$2\sigma_S A_S = 2 p_b b h$$

Then $p_b = \frac{p_a \cdot a}{b}$.

The dependence of the modulus of elasticity, in which $\sigma_\theta = const$ in the entire thickness of the cylinder, loaded with $p_a$ internal pressure and external $p_b$ pressure, has the form [14]:

$$E(r) = E_0 \left(\frac{r}{a} \cdot \frac{A(1-m) - m a \sigma_0}{A(1-m) - m r \sigma_0}\right)^{\frac{1}{1-m}}, \tag{10}$$

Where $E_0 = E(a)$, $A = \frac{(p_b - p_a)ab}{b-a}$, $m = \frac{1-2v}{1-v}$.

Substituting $p_b = \frac{p_a \cdot a}{b}$ in (4.19), we obtain: $E(r) = E_0 \left(\frac{r}{a}\right)^{\frac{1}{1-m}}$.

It remains to determine the required area of the reinforcement, in which $\sigma_{b\theta} = 0$.

$$p_a ah = \sigma_S A_S = (\sigma_{SP} + E_S \varepsilon_\theta|_{r=b}) A_S \tag{11}$$

The circumferential deformation of the concrete on the outer surface can be found as follows:

$$\varepsilon_\theta|_{r=b} = \frac{1}{E_b|_{r=b}}(\sigma_\theta - v(\sigma_r + \sigma_z)) = \frac{1}{E_b|_{r=b}}\{\sigma_\theta - v[\sigma_r + v(\sigma_r + \sigma_\theta)]\}.$$

Given that $\sigma_{b\theta} = 0$ and $\sigma_r|_{r=b} = -p_b$ we get:

$$\varepsilon_\theta|_{r=b} = \frac{p_b}{E_b|_{r=b}}(v + v^2) = \frac{p_a \cdot a}{b \cdot E_b|_{r=b}}(v + v^2) \tag{12}$$

Substituting (12) into (11), after the transformations, we obtain the required area of the reinforcement:

$$A_S = \frac{p_a \cdot a \cdot h}{\sigma_{sp} + \frac{E_S}{E_b|_{r=b}} \cdot \frac{p_a a}{b}(v + v^2)} \tag{13}$$

We note that in formula (13) it depends on the magnitude of the elastic modulus at the outer surface $\left(E_b|_{r=b}\right)$. The smaller $E_b|_{r=b}$, the smaller the required reinforcement.

In Fig. 4 the curve $E(r)$ obtained at is presented $p_a$ = 10 MPa, a = 1 m, b = 1.5 m, $\nu$ = 0.2, $E_0$ = 3.10 * $10^4$, $E_s$ = 2 * $10^5$, $\sigma_{sp}$ = 500 MPa.

In Fig. 5 the stress graph $\sigma_{b\theta}$ is obtained using the finite element method. The deviation $\sigma_{b\theta}$ from zero can be explained by the fact that in calculations the matrix $[B]$, in which the coefficients depend on the radius, was replaced by a matrix $[B]$ in which all values were calculated in the middle of the element to simplify integration.

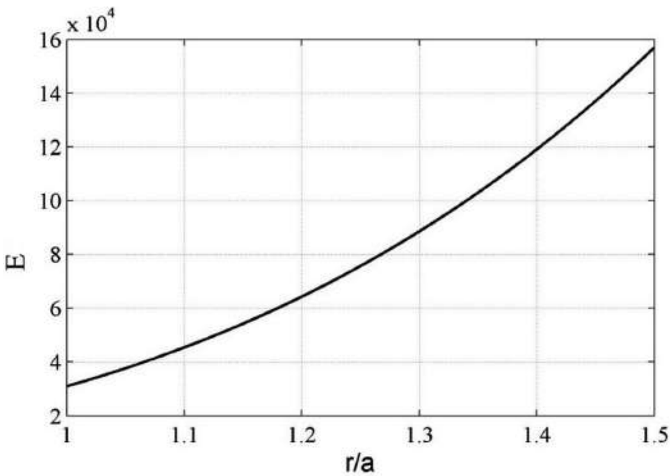


**Fig. 4.** Dependency graph $E(r)$ for the optimal cylinder.

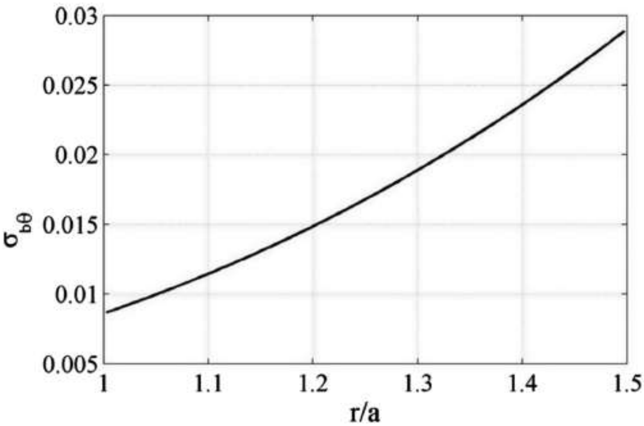


**Fig. 5.** Dependency graph $\sigma_{b\theta}$ for the optimal cylinder.

It should be noted that the decrease in the flow rate of reinforcement for a non-uniform cylinder will be the greater, the larger its thickness, but the modulus of elasticity will also vary over a wider range.

## 4 Conclusions

The solution of the problem of determining the stress-strain state for a reinforced concrete cylinder with unstressed reinforcement and a prestressed cylinder with the location of the stressed reinforcement on the external surface and in the thickness is obtained. By varying the modulus of elasticity, the problem of optimizing the cylinder with external reinforcement was solved, while a reduction in the flow rate of the reinforcement by 10% was achieved. As the thickness of the shell increases, the effect of creating an inhomogeneity increases, but the range of variation of the modulus of elasticity also increases.

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